Cadernos do LOGIS

A branch-and-Benders-cut approach for the single, r and multiple p-hub maximal covering problem with binary and partial coverage criteria

Patrck Doglio, Artur Pessoa and Marcos Roboredo

Volume 2024, Number 6

December, 2024

Preprint

LOGIS Núcleo de Logística Integrada e Sistemas

Universidade Federal Fluminense

A branch-and-Benders-cut approach for the single, r and multiple p-hub maximal covering problem with binary and partial coverage criteria

Patrick Doglio[a], Artur Pessoa[b], Marcos Roboredo[b,*]

[a] *Universidade Federal Fluminense - Pós-Graduação Engenharia de Produção*
Niterói - Brasil
[b] *Universidade Federal Fluminense - Departamento de Engenharia de Produção*
Niterói - Brasil

Abstract

This paper addresses the Uncapacitated Single, r, and Multiple Allocation p-Hub Maximal Covering Problems under both binary and partial coverage criteria. We propose an exact branch-and-Benders-cut algorithm that encompasses all three variants. In this approach, a single branch-and-bound search tree is constructed, and Benders cuts are introduced on demand as the tree is explored. To improve the efficiency of our method, we establish an equivalence between the separation problem and the classical min-cut problem for binary coverage criteria and integer master problem solutions. Through a series of experiments, we demonstrate that our approach outperforms the current state-of-the-art exact algorithms for all three problems. Additionally, we achieve optimal solutions for all instances with up to 50 nodes.

Keywords: branch-and-Benders-cut, Hub location, Maximal Covering.

1. Introduction

Hubs are special facilities that act as switching, transshipment, and sorting points in a many-to-many transportation network. The flows of several origin-destination (O/D) pairs are distributed via inter-hub routes, exploiting economies of scale. This type of distribution system model is widely applied to airlines (Button, 2002), logistic systems (Ishfaq and Sox, 2011), and telecommunication networks (Kim and O'Kelly, 2009).

The Hub Location Problem (HLP) was introduced by O'kelly (1986). In this problem, two decisions are made: the location of a finite number of hubs and the assignment of non-hub points to these located hubs. Campbell (1994) demonstrated that these decisions can be optimized to achieve different objectives related to transportation costs or network coverage. Each HLP can be classified according to its network allocation structure. Three classical classifications are: single allocation (SA) HLP, multiple allocation (MA) HLP, and r allocation (rA) HLP. In a SA-HLP,

*Corresponding author.
Email addresses: patrickdoglio@id.uff.br (Patrick Doglio), arturpessoa@id.uff.br (Artur Pessoa), mcroboredo@id.uff.br (Marcos Roboredo)

each non-hub point is allocated to exactly one hub. In a MA-HLP, each non-hub point can be allocated to all placed hubs if necessary. In a rA-HLP, each non-hub point is allocated to r hubs. The SA-HLP can be viewed as a special case of the rA-HLP when $r = 1$. Additionally, HLPs can be classified as either Capacitated or Uncapacitated. Capacitated HLPs impose a limit on the total flow that can be distributed through a given hub. For further details on HLP definitions and classifications, see Contreras and OKelly (2019) and Farahani et al. (2013).

Among the HLPs studied in Campbell (1994), one is the p-Hub Maximal Covering Problem (pHMCP). In this problem, exactly p hubs are located, and one of the three allocation types (single, multiple, or r) is used. The location and allocation decisions determine a coverage degree for each O/D pair, which is either 1 (totally covered) or 0 (not covered) based on its service cost. This coverage criterion is referred to as the Binary Criterion (BC). The objective of the pHMCP is to maximize the sum of the flows associated with totally covered O/D pairs.

Peker and Kara (2015) proposed a generalization of the pHMCP that introduces the partial coverage criterion (PC) for an O/D pair. This variant considers a discrete set of coverage degrees L ranging from 0 to 1, and a non-increasing function that associates the service cost of an O/D pair with a value from L. The goal is to maximize the sum of the flow of each O/D pair, weighted by its coverage degree. When $L = \{0, 1\}$, this problem reduces to the pHMCP with the binary coverage criterion.

This paper addresses the Uncapacitated Single, r, and Multiple Allocation p-Hub Maximal Covering Problems with binary and partial coverage criteria. These problems are denoted as USApHMCP, UrApHMCP, and UMApHMCP, respectively.

We propose an exact Benders Decomposition (BD) algorithm that covers the USApHMCP, UrApHMCP, and UMApHMCP with binary and partial coverage criteria. Specifically, our algorithm employs an integer programming (IP) formulation, where Benders cuts are used both as lazy constraints and to strengthen the linear relaxation. To improve the computational efficiency of the proposed approach, we demonstrate that the separation problems are equivalent to classical min-cut problems for the binary coverage criterion and for integer solutions. Hence, we employ a specialized algorithm for these cases.

BD algorithms that utilize a single search tree are referred to in the literature as branch-and-Benders-cut (BBC) algorithms (Li and Aneja, 2020; Moreno et al., 2019; Errico et al., 2017; Gendron et al., 2016). BBC algorithms have shown superior effectiveness compared to the traditional Benders approach, which requires solving a mixed-integer master problem at each iteration. Similar algorithms for other HLPs were proposed by Ghaffarinasab and Kara (2019) and Ghaffarinasab (2020).

To assess the effectiveness of our proposed BBC algorithm, we compare it with state-of-the-art exact formulations for the USApHMCP, UrApHMCP, and UMApHMCP, as proposed by Peker and Kara (2015), Stančić et al. (2022), and Janković et al. (2017), respectively. The results indicate that our method outperforms the existing approaches for most instances, particularly for the largest ones. Furthermore, we present optimal solutions for all AP instances with up to 50 nodes for the first time.

The remainder of this paper is organized as follows. Section 2 presents a literature review of the uncapacitated pHMCP and its variations. Section 3 formally defines the USApHMCP, UrApHMCP, and UMApHMCP with binary and partial coverage criteria and presents the current state-of-the-art exact formulations for these problems. In Section 4, we describe the proposed exact BBC algorithm. Section 5 presents the computational results of the proposed method. Finally, Section 6 summarizes the conclusions and suggests future research directions.

2. Literature Review

In this section, we provide a literature review on the uncapacitated pHMCP and its variations. The problems discussed differ in terms of the type of allocation (SA, MA, or rA) and the type of coverage criteria (BC or PC) considered.

The early papers on the pHMCP focused solely on binary coverage criteria. This problem was introduced by Campbell (1994), where the author presented MILP formulations with $O(n^4)$ variables and constraints for different allocation types. Later, Weng et al. (2006) proposed two metaheuristics and a MILP formulation with $O(n^2)$ variables and constraints for the UMApHMCP. Qu and Weng (2009) also addressed the UMApHMCP, proposing a path-relinking approach. Hwang and Lee (2012) introduced an MILP formulation with $O(n^4)$ variables and constraints for the USApHMCP.

Peker and Kara (2015) introduced the concept of partial coverage for the pHMCP. The authors proposed MILP formulations for both the UMApHMCP and USApHMCP that are valid for both binary and partial coverage criteria. For the binary coverage criterion, their formulations were shown to outperform those previously proposed in the literature. Janković et al. (2017) also addressed the UMApHMCP and USApHMCP with both binary and partial coverage criteria. The authors proposed heuristic algorithms and MILP formulations for these problems. For the UMApHMCP, their new formulation outperformed the one proposed by Peker and Kara (2015), but this was not the case for the USApHMCP. Recently, Stančić et al. (2022) proposed two heuristic algorithms and an MILP formulation for the UrApHMCP that can be applied to both binary and partial coverage criteria.

Although recent, some papers focused solely on the pHMCP with binary coverage criteria. In this context, Silva and Cunha (2017) proposed a tabu search algorithm for the USApHMCP. Janković and Stanimirović (2017) addressed the UrApHMCP for the first time, proposing an MILP formulation with $O(n^4)$ variables and a variable neighborhood search algorithm for the problem with the binary coverage criterion.

The pHMCP is important in its own right. However, it is noteworthy that this problem models the subproblems of certain competitive HLPs, such as those studied by Ghaffarinasab et al. (2018) and de Araújo et al. (2020).

	Allocation			Cov. criterion		Algorithm	
Reference	MA	SA	rA	Binary	Partial	Exact	Heuristic
Campbell (1994)	✓	✓		✓		✓	
Weng et al. (2006)	✓			✓		✓	✓
Qu and Weng (2009)	✓			✓			✓
Hwang and Lee (2012)		✓		✓		✓	✓
Peker and Kara (2015)	✓	✓		✓	✓	✓	
Janković et al. (2017)	✓	✓		✓	✓	✓	✓
Janković and Stanimirović (2017)			✓	✓		✓	✓
Silva and Cunha (2017)		✓		✓			✓
Stančić et al. (2022)			✓	✓	✓	✓	✓
This Paper	✓	✓	✓	✓	✓	✓	

Table 1: Summary of the literature on the uncapacitated pHMCP.

This paper proposes an exact approach for the UrApHMCP, UMApHMCP, and USApHMCP that is valid for both binary and partial coverage criteria. For these problems, the state-of-the-art exact formulations in the literature are proposed by respectively Stančić et al. (2022) (UrApHMCP), Janković et al. (2017) (UMApHMCP), and Peker and Kara (2015) (USApHMCP). Table 1 classifies the references on the uncapacitated pHMCP according to the type of allocation, coverage criterion, and algorithm (exact or heuristic) considered.

3. Problem definitions and Formulations

This section formally defines the UrApHMCP with binary and partial coverage criteria. This definition can be extended to the USApHMCP by setting $r = 1$ and to the UMApHMCP by setting $r = p$, respectively. Moreover, this section presents the state-of-the-art exact formulations from the literature for the three problems.

Let $G = (N, A)$ be a complete directed graph, where N and A represent the set of vertices and arcs, respectively, with $N = \{1, \ldots, n\}$. Let $H \subseteq N$ be a set of potential hubs. Each arc $(i, j) \in A$ represents an O/D pair and is associated with a cost $d_{ij} > 0$ and a flow $w_{ij} \geq 0$. The problem involves two decisions: the first is to choose p vertices from H to locate hubs, and the second is to allocate each vertex in N to at most r located hubs ($r \leq p$).

After the location and allocation decisions, the flow w_{ij} is transported from i to j via a single path $i \to k \to m \to j$, where k and m are located hubs, and i and j are allocated to k and m, respectively. The hubs k and m may be distinct vertices, or they may coincide. The service cost for the transportation along this path is given by:

$$c_{ij}^{km} = \chi d_{ik} + \alpha d_{km} + \delta d_{mj}, \tag{1}$$

where $\chi \geq 1$, $\alpha \in [0, 1]$, and $\delta \geq 1$ are unitary rates for collecting, transferring, and distributing flow items, respectively. Equation (1) is also valid when $k = m$, indicating that the transportation occurs through a single hub.

The problem also considers a discrete set L of coverage degrees with values between 0 and 1, and a non-increasing function $f : \mathbb{R} \rightarrow L$ that associates each service cost with a coverage degree $l \in L$. Thus, each O/D pair is assigned a coverage degree from L. When $L = \{0, 1\}$, the coverage criterion is binary. Otherwise, the coverage criterion is partial. The objective is to maximize the sum of each O/D pair flow, weighted by its associated coverage degree.

Sections 3.1, 3.2, and 3.3 present the state-of-the-art MILP formulations from the literature for the UrApHMCP, UMApHMCP, and USApHMCP, respectively.

3.1. State-of-art exact MILP formulation for the UrApHMCP

The state-of-the-art MILP formulation for the UrApHMCP was proposed by Stančić et al. (2022). Let us define $L^k_{ij} = \{l \in L \mid$ there exists at least one $m \in H$ such that $f(c^{km}_{ij}) = l\}$, and $G^{kl}_{ij} = \{m \in H \mid f(c^{km}_{ij}) = l\}$. We also define the following decision variables:

- z_{ij} non-negative continuous variable indicating the degree of coverage for the O/D pair (i, j).
- s_k binary variable indicating if a hub is located at vertex k ($s_k = 1$) or not ($s_k = 0$).
- x_{ik} binary variable indicating if vertex i is allocated to the placed hub k ($x_{ik} = 1$) or not ($x_{ik} = 0$).
- w_{ijkl} binary variable indicating if vertex i is allocated to the placed hub k, vertex j is allocated to some $m \in G^{kl}_{ij}$, and the O/D pair (i, j) is covered with degree l.

The complete literature formulation follows.

$$\max \sum_{i \in N} \sum_{j \in N} w_{ij} z_{ij} \tag{2a}$$

$$\text{s.t.} \sum_{k \in H} s_k = p \tag{2b}$$

$$\sum_{k \in H} x_{ik} \leq r, \qquad i \in N \tag{2c}$$

$$x_{ik} \leq s_k, \qquad i \in N, k \in H \tag{2d}$$

$$w_{ijkl} \leq x_{ik}, \qquad i, j \in N, k \in H, l \in L^k_{ij} \tag{2e}$$

$$w_{ijkl} \leq \sum_{m \in G^{kl}_{ij}} x_{jm}, \qquad i, j \in N, k \in H, l \in L^k_{ij} \tag{2f}$$

$$\sum_{k \in H} \sum_{l \in L^k_{ij}} w_{ijkl} \leq 1, \qquad i, j \in N \tag{2g}$$

$$z_{ij} \leq \sum_{k \in H} \sum_{l \in L^k_{ij}} l \times w_{ijkl}, \qquad i, j \in N \tag{2h}$$

$$z_{ij} \geq 0, \qquad i,j \in N \tag{2i}$$
$$x_{ik} \in \{0,1\}, \qquad i \in N, k \in H \tag{2j}$$
$$w_{ijkl} \in \{0,1\}, \qquad i,j \in N, k \in H, l \in L_{ij}^{k} \tag{2k}$$
$$s_k \in \{0,1\}, \qquad k \in H \tag{2l}$$

The objective function (2a) aims to optimize the total sum of flows weighted by their degrees of coverage. Constraint (2b) ensures that exactly p hubs are located. Constraints (2c) enforce the r-allocation strategy, where for $r = 1$, the problem corresponds to the USApHMCP. Constraints (2d) ensure that vertices are only allocated to hubs that are actually located. Constraints (2e) and (2f) ensure that w_{ijkl} can only be equal to one if $x_{ik} = 1$ and there is at least one $m \in G_{ij}^{kl}$ such that $x_{jm} = 1$. Constraint (2g) ensures that for each O/D pair (i,j), there is at most one covering path established. Finally, constraint (2h) calculates the coverage degree for each O/D pair (i,j). The remaining constraints define the variable domains.

3.2. State-of-art exact MILP formulation for the UMApHMCP

The state-of-the-art MILP formulation for the UMApHMCP was proposed by Janković et al. (2017) as follows:

$$\max \text{(2a)} \tag{3a}$$
$$\text{s.t. (2b), (2g), (2h), (2i), (2j), (2k), (2l)} \tag{3b}$$
$$w_{ijkl} \leq s_k, \qquad i,j \in N, k \in H, l \in L_{ij}^{k} \tag{3c}$$
$$w_{ijkl} \leq \sum_{m \in G_{ij}^{kl}} s_m, \qquad i,j \in N, k \in H, l \in L_{ij}^{k} \tag{3d}$$

Formulation (3) is similar to Formulation (2). The main difference is that Formulation (3) does not consider the allocation variables x, as each non-hub point is allocated to all placed hubs in the UMApHMCP.

3.3. State-of-art exact MILP formulation for the USApHMCP

The state-of-the-art MILP formulation for the USApHMCP was proposed by Peker and Kara (2015). This formulation is valid for both binary and partial coverage criteria. It uses a constant λ_{ij} for each O/D pair (i,j), calculated as $\lambda_{ij} = \max_{k,m \in H} f(c_{ij}^{km})$. The formulation is as follows:

$$\max \text{(2a)} \tag{4a}$$
$$\text{s.t. (2b), (2d), (2j), (2l)} \tag{4b}$$
$$\sum_{k \in H} x_{ik} = 1, \qquad i \in N \tag{4c}$$

$$z_{ij} \leq \sum_{k \in H} f(c_{ij}^{km}) x_{ik} + \lambda_{ij} \left(1 - x_{jm}\right), \qquad i, j \in N, m \in H \tag{4d}$$

Constraint (4c) ensures the single allocation strategy. For each O/D pair (i, j), Constraints (4d) ensure that the variable z_{ij} is equal to $f(c_{ij}^{km})$ for the hubs k and m such that $x_{ik} = x_{jm} = 1$.

4. Proposed Resolution Method

In this section, we present a Branch-and-Cut (BBC) algorithm that encompasses the UrApHMCP, UMApHMCP, and USApHMCP, with both binary and partial coverage criteria.

4.1. Main Formulation

To present the main formulation used in the proposed algorithm for the UrApHMCP and USApHMCP, we define the following additional binary variables:

$z_{ij}^l \rightarrow$ binary variable indicating if the O/D pair (i, j) is covered with degree of coverage $l \in L$.

The formulation is as follows:

$$\max \sum_{i \in N} \sum_{j \in N} \sum_{l \in L} l \times w_{ij} \times z_{ij}^l \tag{5a}$$

$$\text{s.t. } (2b), (2c), (2d), (2j) \tag{5b}$$

$$\sum_{l \in L} z_{ij}^l = 1, \qquad i, j \in N \tag{5c}$$

$$\sum_{l \in L} z_{ij}^l \pi_{ijl} + \sum_{k \in H} x_{ik} \theta_{ijk} + \sum_{m \in H} x_{jm} \rho_{ijm} \geq 0, \qquad i, j \in N, \ (\pi_{ij}, \theta_{ij}, \rho_{ij}) \in R_{ij} \tag{5d}$$

$$z_{ij}^l \in \{0, 1\}, \qquad i, j \in N, l \in L, \tag{5e}$$

In Formulation (5), the objective function (5a) represents the sum of the coverage levels of all O/D pairs, weighted by their flows. Constraints (5c) ensure that only a single degree of coverage is assigned to each O/D pair (i, j). Finally, Constraints (5d) involve constants R_{ij}, which are sets of Benders-cut-coefficient vectors associated with the O/D pair (i, j). These vectors are derived from the Benders decomposition of an extended formulation that will be described later. Constraints (5d) ensure the consistency of the z_{ij}^l and x_{ik} variables.

Note that the consistency between the x_{ik} and z_{ij}^l variables is ensured only by Constraints (5d). Thus, (5d) should be treated as lazy constraints. The rationale behind this modeling approach lies in the fact that the relationship between each O/D pair and the corresponding hub pair is better represented by a four-index variable that explicitly defines the specific origin, destination, and hubs used in the transportation. However, while these variables lead to a much stronger formulation, their direct use is prohibitive due to the large number of such variables. Throughout this paper, we refer to any linear relaxation of (5) with a subset of the Constraints (5d) and possibly including additional branching constraints as the Node Master Problem (NMP), since the BBC algorithm needs to solve one or more problems of this type for each branch-and-bound node.

4.2. Extended formulation

The extended formulation used as a starting point to develop our algorithm is the one proposed by Kara and Tansel (2003) for a problem that is quite similar to the USApHMCP with binary coverage criterion. Instead of maximizing the total covering through placing p hubs, one seeks to minimize the number of placed hubs needed for a complete covering. The formulation of Kara and Tansel (2003) can be immediately adapted to the USApHMCP with binary coverage criterion as presented next. For that, we define the following y variables:

$y_{ijkm} \rightarrow$ continuous variable indicating the fraction of the flow w_{ij} that is transported through the path $i \rightarrow k \rightarrow m \rightarrow j$, $i, j \in N$, $k, m \in H$. The formulation for the USApHMCP with binary coverage criterion adapted from Kara and Tansel (2003) follows.

$$\max \sum_{i \in N} \sum_{j \in N} \sum_{k,m \in H} w_{ij} a_{ijkm} y_{ijkm} \tag{6a}$$

$$s.t. \ (2b), (4c), (2d), (2j) \tag{6b}$$

$$\sum_{m \in H} y_{ijkm} = x_{ik} \qquad i, j \in N, k \in H \tag{6c}$$

$$\sum_{k \in H} y_{ijkm} = x_{jm} \qquad i, j \in N, m \in H \tag{6d}$$

$$y_{ijkm} \geq 0 \qquad i, j \in N, k, m \in H \tag{6e}$$

where

$$a_{ijkm} = \begin{cases} 1 & \text{if the vertices } k \text{ and } m \text{ cover the O/D pair } (i, j), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The objective function (6a) aims to maximize the total sum of flows weighted by their degrees of coverage. Constraints (6c) and (6d) result from the linearization employed by Kara and Tansel (2003) over a quadratic formulation where each product of variables $x_{ik} \times x_{jm}$ is replaced with y_{ijkm}. They ensure the consistency between the two types of variables.

4.3. Extending for r-allocation and partial coverage criterion

In order to extend the Formulation (6) to the other variants addressed in this paper, we relax Constraints (6c) and (6d) to inequalities, and add an additional equality.

$$\sum_{m \in H} y_{ijkm} \leq x_{ik} \qquad i,j \in N, k \in H \tag{8}$$

$$\sum_{k \in H} y_{ijkm} \leq x_{jm} \qquad i,j \in N, m \in H \tag{9}$$

$$\sum_{k,m \in H} y_{ijkm} = 1 \qquad i,j \in N \tag{10}$$

The following proposition shows that this change cannot have a negative impact on the quality of the continuous relaxation of (6b)-(6e).

Proposition 1. *Any solution to* (4c)*,* (8)*,* (9)*, and* (10) *also satisfies* (6c) *and* (6d)*.*

Proof. We only prove that (6c) is satisfied under the stated conditions because the case (6d) is analogous. Since (8) is assumed to be satisfied, we only need to prove that $\sum_{m \in H} y_{ijkm} \geq x_{ik}$ for all $i, j \in N, k \in H$. We have that

$$\begin{aligned} \sum_{m \in H} y_{ijkm} &= 1 - \sum_{k' \in H \setminus \{k\}} \sum_{m \in H} y_{ijk'm} \\ &\geq 1 - \sum_{k' \in H \setminus \{k\}} x_{ik'} \\ &= x_{ik}. \end{aligned}$$

The first equality, the inequality and the last equality are a consequences of (10), (8) and (4c), respectively. □

Note that the previous proof is also valid if (4c) is replaced by

$$\sum_{k \in H} x_{ik} \leq 1, \qquad i \in N. \tag{11}$$

Thus, by replacing again (11) with (2c), we obtain a formulation that naturally extends the formulation of Kara and Tansel (2003) also for the UrApHMCP with the binary coverage criterion.

In order to also treat the partial coverage criterion, one only needs to replace a_{ijkm} with $f(c_{ij}^{km})$ in (6a). Instead, to make the formulation even stronger, we replace the z_{ij} variables with the z_{ij}^l variables defined in Subsection 4.1. For that, let us define the constants a_{ijkm}^l ($i, j \in N$, $k, m \in H$, $l \in L$) as follows:

$$a^l_{ijkm} = \begin{cases} 1 & \text{if the vertices } k \text{ and } m \text{ cover the O/D pair } (i,j) \text{ with degree} \\ & \text{of coverage } l \ (f(c^{km}_{ij}) = l) \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

The resulting formulation follows.

$$\max \sum_{i\in N}\sum_{j\in N}\sum_{l\in L} l \times w_{ij} \times z^l_{ij} \tag{13a}$$

$$s.t. \sum_{k\in H} s_k = p \tag{13b}$$

$$\sum_{k\in H} x_{ik} \le r \qquad i \in N \tag{13c}$$

$$x_{ik} \le s_k, \qquad i \in N, k \in H \tag{13d}$$

$$\sum_{m\in H} y_{ijkm} \le x_{ik} \qquad i,j \in N, k \in H \tag{13e}$$

$$\sum_{k\in H} y_{ijkm} \le x_{jm} \qquad i,j \in N, m \in H \tag{13f}$$

$$\sum_{l\in L} z^l_{ij} = 1 \qquad i,j \in N \tag{13g}$$

$$\sum_{k,m\in H} a^l_{ijkm} y_{ijkm} = z^l_{ij} \qquad i,j \in N, l \in L \tag{13h}$$

$$z^l_{ij} \in \{0,1\} \qquad i,j \in N, l \in L \tag{13i}$$

$$x_{ik} \in \{0,1\}, \qquad i \in N, k \in H \tag{13j}$$

$$y_{ijkm} \ge 0 \qquad i,j \in N, k,m \in H \tag{13k}$$

In this formulation, (13a), (13b), (13c), (13d), (13e), (13f), and (13g) are identical to (5a), (2b), (2c), (2d), (8), (9), and (5c) respectively. Constraints (13h) ensure the definition of the z^l_{ij} variables. Note that Constraints (10) become redundant since it can be obtained from (13g) by replacing z^l_{ij} with the left-hand side of (13h), and using the fact that $\sum_{l\in L} a^l_{ijkm} = 1$ by definition.

4.4. Benders Cuts

Now, we define Constraints (5d) as a result of a BD applied to the linear relaxation of (13). For fixed $\bar{x}$ and $\bar{z}$, the primal Benders subproblem becomes:

$$\max 0 \tag{14a}$$

$$s.t. \ (13k) \tag{14b}$$

$$\sum_{k,m \in H} a^l_{ijkm} y_{ijkm} = \bar{z}^l_{ij} \qquad i,j \in N, l \in L \tag{14c}$$

$$\sum_{m} y_{ijkm} \leq \bar{x}_{ik}, \qquad i,j \in H, k \in H \tag{14d}$$

$$\sum_{k} y_{ijkm} \leq \bar{x}_{jm}, \qquad i,j \in H, m \in H \tag{14e}$$

Since the objective function is zero, only feasibility cuts are needed. Let π_{ijl}, θ_{ijk} and ρ_{ijm} be the dual variables for respectively (14c), (14d) and (14e). The Dual Subproblem (DS) is defined separately for each O/D pair (i, j) as follows:

$$\min \sum_{l \in L} \bar{z}^l_{ij} \pi_{ijl} + \sum_{k \in H} \bar{x}_{ik} \theta_{ijk} + \sum_{m \in H} \bar{x}_{jm} \rho_{ijm} \tag{15a}$$

$$s.t. \sum_{l \in L} a^l_{ijkm} \pi_{ijl} + \theta_{ijk} + \rho_{ijm} \geq 0 \qquad k, m \in H \tag{15b}$$

$$\pi_{ijl} \text{ free} \qquad l \in L \tag{15c}$$

$$\theta_{ijk} \geq 0 \qquad k \in H \tag{15d}$$

$$\rho_{ijm} \geq 0 \qquad m \in H \tag{15e}$$

The DS is always feasible since 0 is a feasible solution. Thus, a feasibility cut is generated only if DS is unbounded. In this case, let $(\pi^*_{ijl},\ \theta^*_{ijk},\ \rho^*_{ijm})$ be an extreme ray that makes (15a) negative. The following cut is added to the master problem:

$$\sum_{l \in L} z^l_{ij} \pi^*_{ijl} + \sum_{k \in H} x_{ik} \theta^*_{ijk} + \sum_{m \in H} x_{jm} \rho^*_{ijm} \geq 0.$$

Defining R_{ij} as the set of extreme rays of feasible set of (15), for each $i, j \in N$, we obtain Constraints (5d).

4.5. Interpreting the DS as the multicommodity flow problem

Now we show that the DS problem for a fixed O/D pair (i, j) can be viewed as the multicommodity flow problem. Let (i, j) be an O/D pair and let $G_{ij} = (V_{ij}, A_{ij})$ be a graph, where $V_{ij} = \{v_0, ..., v_{2|H|+1}\}$ and $A_{ij} = \{(v_0, v_k) : k \in H\} \cup \{(v_k, v_{|H|+m}) : k, m \in H\} \cup \{(v_{|H|+m}, v_{2|H|+1}) : m \in H\}$. y_{ijkm} in the primal subproblem can be viewed as the amount of flow of commodity $l = f(c^{km}_{ij})$ that passes through the path $(v_0, v_k, v_{|H|+m}, v_{2|H|+1})$ in G_{ij}. In this case, (13k), (14c), (14d) and (14e) are feasible if and only if there is a multicommodity flow from v_0 to $v_{2|H|+1}$ in G_{ij} where the total flow of commodity $l \in L$ is $\bar{z}^l_{ij}$ (Constraints (14c)), and the capacities $\bar{x}_{ik}$ and $\bar{x}_{jm}$ on arcs (v_0, v_k) and $(v_{|H|+m}, v_{2|H|+1})$ are respected, respectively (Constraints (14d) and (14e), respectively).

4.5.1. Particular Cases

Given a fixed O/D pair (i, j), for the binary coverage criteria or for the case where all values of $\bar{z}^l_{ij}$ are integer, there is at most one $l \in L \setminus \{0\}$ such $\bar{z}^l_{ij} > 0$. This condition is satisfied because for the binary coverage criteria we have by definition only one value of l different from zero since $L = \{0, 1\}$ and for the second case when all values of $\bar{z}^l_{ij}$ are integer, the Constraints (5c) ensure that condition. Thus, for the particular cases described before, let $l \in L$ such $\bar{z}^l_{ij} > 0$, the Constraints (13k), (14c), (14d) and (14e) are feasible if and only if there is single commodity flow with value $\bar{z}^l_{ij}$ from v_0 to $v_{2|H|+1}$ in $G^l_{ij} = (V_{ij}, A^l_{ij})$, where $A^l_{ij} = A_{ij} \setminus \{(v_k, v_{|H|+m}) : k, m \in H, a^l_{ijkm} = 0\}$, respecting the capacities on arcs $\bar{x}_{ik}$ and $\bar{x}_{jm}$ defined above. In the remainder of this subsection, we show how to check it and extract valid cuts in the form of (5d) in case of infeasibility.

For the sake of readability, let us present our techniques through a toy example. For this example, suppose that $H = \{1, 2, 3\}$, and that $(1, 3), (2, 1), (2, 2)$ and $(3, 3)$ are the hub pairs that cover the O/D pair (i, j) with degree l ($a^{13}_{ij} = a^{21}_{ij} = a^{22}_{ij} = a^{33}_{ij} = 1$). In this case, we have $V_{ij} = \{v_0, ..., v_7\}$ and $A^l_{ij} = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_4, v_7), (v_5, v_7), (v_6, v_7), (v_1, v_6), (v_2, v_4), (v_2, v_5), (v_3, v_6)\}$. The Figure 1 illustrates the graph for the proposed example.

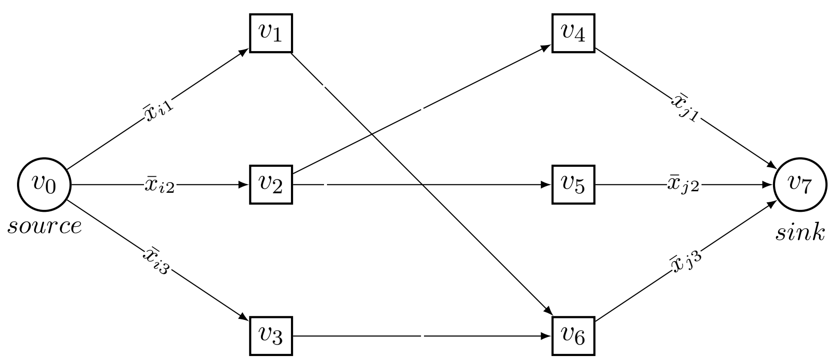

Figure 1: G^l_{ij} for the proposed toy example.

The graph G^l_{ij} is built in a way that each path from v_0 to v_7 is associated with exactly one subproblem variable y_{ijkm} such that $f(c^{km}_{ij}) = l$, and the value of each such variable determines the amount of flow passing through the corresponding path. For example, for the graph of Figure 1, the path $v_0 \rightarrow v_2 \rightarrow v_4 \rightarrow v_7$ is associated with y_{ij21}. Thus, the single commodity flow mentioned above is composed by all the flows associated to these variables. Now we show how to extract a valid inequality from G^l_{ij} when such a flow does not exist. Let $S \subset V_{ij}$ be an $(v_0, v_{2|H|+1})-$cut for the graph G^l_{ij}. In other words, S is a subset of vertices such that $v_0 \in S$ and $v_{2|H|+1} \notin S$. An $(v_0, v_{2|H|+1})-$cut has the property that any path from v_0 to $v_{2|H|+1}$ uses at least one arc from $\delta_+(S) = A^l_{ij} \cap (S \times (V_{ij} \setminus S))$. For example, for the graph from the Figure 1, let $S = \{v_0, v_2, v_4, v_5\}$ be a $(v_0, v_7)-$cut. In this case, $\delta_+(S) = \{(v_0, v_1), (v_0, v_2), (v_4, v_7), (v_5, v_7)\}$. Figure 2 illustrates the set S and the arcs in $\delta_+(S)$, marked in red.

As any path from v_0 to v_7 has to use at least one marked arc, we can affirm that the amount of flow $\bar{z}^l_{ij}$ that goes from v_0 to v_7 cannot be greater than $\bar{x}_{i1} + \bar{x}_{i3} + \bar{x}_{j1} + \bar{x}_{j2}$, and, since the graph

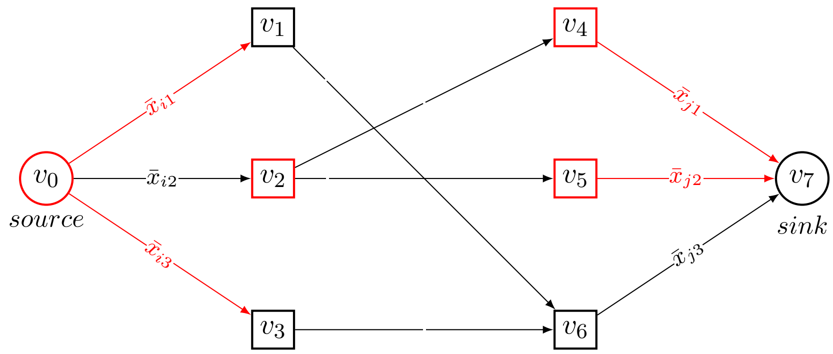

Figure 2: An $(v_0, v_7)-$ cut in G^l_{ij} for the proposed toy example.

structure is maintained as the solution of the master problem changes, the following inequality should be satisfied by any master problem solution:

$$z^l_{ij} \le x_{i1} + x_{i3} + x_{j1} + x_{j2}. \tag{16}$$

Other valid inequalities can be generated in this way by taking different $(v_0, v_7)-$cuts, and taking the sum of the capacities of the corresponding arcs as the right-hand side. We highlight that if we take an $(v_0, v_7)-$cut S where $v_h \in S$ and $v_{h'} \notin S$, with $h \le |H|$, $|H| < h' \le 2|H|$ and $a^l_{ijhh'} = 1$ then a large constant appears on the right-hand side of the inequality since the middle arcs has an unlimited capacity. Consequently, the inequality can not be violated. Let S^l_{ij} be the set of $(v_0, v_{2|H|+1})$-cut without the previous characteristic. Mathematically, we have

$$\begin{aligned} S^l_{ij} \quad = \quad & \{S|S \text{ is a } (v_0, v_{2|H|+1})\text{-cut for } G^l_{ij} \text{ and} \\ & v_k \in S \Rightarrow v_{|H|+m} \in S \text{ whenever } a^l_{ijkm} = 1\} \end{aligned} \tag{17}$$

The inequalities in a generic way are defined as

$$z^l_{ij} \le \sum_{h \in H: k_h \notin S} x_{ih} + \sum_{h \in H: m_h \in S} x_{jh}, \;\; i, j \in N, l \in L, S \in S^l_{ij}, \tag{18}$$

which corresponds to (5d) by setting $\pi_{ijl} = -1$, $\theta_{ijh} = 1$ if $k_h \notin S$ and zero otherwise, and $\rho_{ijh} = 1$ if $m_h \in S$ and zero otherwise, for all $h \in H$. By the results shown in this section, if one finds a given solution $\bar{x}$, $\bar{z}$ to NMP that meets this particular case in a course of the BBC algorithm, we do not need to solve the Formulation (15) for each O/D pair (i, j). Instead, we can find the $(v_0, v_{2|H|+1})$-cut S that minimize the right side of (18) for each O/D pair (i, j) and for the single value of l such that $\bar{z}^l_{ij} > 0$ considering the fixed values $\bar{x}$. This problem is a equivalent to the classical min-cut problem on G^l_{ij}. If the optimal value of this problem is smaller than $\bar{z}^l_{ij}$, then we add to the NMP the constraint $z^l_{ij} \le \sum_{h \in H: k_h \notin S} x_{ih} + \sum_{h \in H: m_h \in S} x_{jh}$ as a benders-like cut.

4.6. The proposed BBC algorithm

Now, we present the proposed exact BBC algorithm to solve the Main Formulation (5). In our approach, the problem is solved on a single B&B tree, and a cut (5d) is added for each O/D pair using the lazy constraint callback function available in some commercial solvers, such as CPLEX. We continue adding cuts to fractional solutions only while the gap of the tree is greater than a given parameter ϵ. For a fixed NMP solution ($\bar{x}$, $\bar{z}$), we attempt to find a cut of type (5d) for each O/D pair (i, j) by solving the Formulation (15) if the coverage criterion is partial and $\bar{x}$ is fractional. Otherwise, we seek to find a cut of type (18) for each O/D pair (i, j) and for the single value of l such that $z^l_{ij} > 0$, by solving the min-cut problem on G^l_{ij}. Once the CPLEX solver applies its own preprocessing and is able to leave the root node based on criteria beyond our control, we propose a procedure to calculate the correct upper bound at the root node generated by the proposed method. The steps of the proposed BBC algorithm are shown in Algorithm 1.

The algorithm 1 begins at line 25 by calling the procedure `RootNode` proposed in this paper to calculate the correct upper bound at the root node. This procedure is described in Algorithm 2 and consists of two steps. In the first step, we solve the linear relaxation of Formulation (5) where the constraints (5d) are replaced with the constraints in the set CUTS, which is initially empty (line 6). In the second step (lines 7 - 24), we look for violated constraints based on the solution found. When the type of coverage is binary, we look for violated constraints (18) by solving the minimum $(v_0, v_{2|H|+1})$-cut problem on G^l_{ij} (procedure MINCUT of line 10) for each O/D pair (i, j) and for each coverage degree l such that $\bar{z}^l_{ij} > 0$. When the type of coverage is partial, we look for violated constraints (5d) by solving the linear programming formulation (15) for each O/D pair (i, j). If this formulation is unbounded, we have a violated cut (5d) by taking an extreme ray from DS (lines 17 and 18). The violated constraints are added to the set CUTS, and steps one and two are executed again. When no violated constraints are found, we delete from CUTS the constraints whose last dual value is null and stop the procedure.

In line 26 of Algorithm 1, we solve Formulation (5) where the constraints (5d) are replaced with the constraints in the set CUTS returned by the `RootNode` procedure. Additionally, we use the `CALLBACK()` procedure as a callback function to add lazy constraints on demand. This callback is called at each B&B tree node, where the value of the current relaxed solution $(\bar{x}, \bar{z})$ is given. In the `CALLBACK()` procedure, we first check for each O/D pair (i, j) if its corresponding variables $\bar{x}$ are all integers (line 3). If so, we check if there is a coverage degree l such that $z^l_{ij} > 0$ (line 4). If this is the case, a set S is found by solving the minimum $(v_0, v_{2|H|+1})$-cut problem on G^l_{ij} (procedure MINCUT of line 5). If S is associated with a violated constraint (18), the corresponding cut is added to the formulation (lines 6 - 8). When the values of $\bar{x}$ associated with the O/D pair (i, j) are not all integers, we only attempt to find violated constraints to add to the NMP if the relative percentage difference between the best lower and upper B&B tree bounds (gap) is greater than a prespecified value $\epsilon \in [0, 1]$ (line 11). When this happens, if the binary coverage criterion is in use, we try to find a violated constraint (18) for each degree of coverage $l \in L \setminus \{0\}$ by executing steps 4 - 9 (line 13). Otherwise, we solve Formulation (15) (line 15). If this formulation is unbounded, we add a violated cut (5d) by taking an extreme ray from DS (lines 17 and 18).

Algorithm 1 Porposed BBC algorithm

1: **procedure** CALLBACK(NMP,$\bar{x}, \bar{z}$)
2:     **for** each O/D pair (i, j) **do**
3:         **if** $\bar{x}_{ik}$ and $\bar{x}_{jm}$ are all integer, for $k, m \in H$ **then**
4:             **for** each $l \in L \setminus \{0\}$ with $\bar{z}^l_{ij} > 0$ **do**
5:                 $S \leftarrow$ MINCUT(G^l_{ij})
6:                 **if** $\sum_{h \in H: k_h \notin S} \bar{x}_{ih} + \sum_{h \in H: m_h \in S} \bar{x}_{jh} < \bar{z}^l_{ij}$ **then**
7:                     Add $z^l_{ij} \leq \sum_{h \in H: k_h \notin S} x_{ih} + \sum_{h \in H: m_h \in S} x_{jh}$ to NMP.
8:                 **end if**
9:             **end for**
10:         **else**
11:             **if** $gap > \epsilon$ **then**
12:                 **if** the coverage criterion is binary **then**
13:                     Execute the steps 4 - 9
14:                 **else**
15:                     Solve the DS Formulation (15)
16:                     **if** DS is unbounded **then**
17:                         $(\pi^*_{ijl}, \theta^*_{ijk}, \rho^*_{ijm}) \leftarrow$ extreme ray that violates (15a)
18:                         Add $\sum\limits_{l \in L} z^l_{ij} \pi^*_{ijl} + \sum\limits_{k \in H} x_{ik} \theta^*_{ijk} + \sum\limits_{m \in H} x_{jm} \rho^*_{ijm} \geq 0$ to NMP.
19:                     **end if**
20:                 **end if**
21:             **end if**
22:         **end if**
23:     **end for**
24: **end procedure**
25: CUTS $\leftarrow$ Procedure RootNode
26: Solve a version of (5) where the constraints (5d) are replaced with the constraints in CUTS calling CALLBACK() to add cuts in each BBC node.
27: **Return** The optimal solution found by the solver.

4.7. Dealing with the UMApHMCP

The proposed algorithm can also be applied to UMApHMCP by setting $r = p$. However, for this variant, it is unnecessary to consider the allocation variables x, since each non-hub point is allocated to all placed hubs in UMApHMCP. Therefore, we propose the following simple modifications to Algorithm (1) when dealing with UMApHMCP. We omit the x variables and the constraints (2c) and (2d) in formulation (5), and replace each variable x_{ik} and x_{jm} with s_k and s_m, respectively, in constraints (5d) and (18).

5. Computational Experiments

We now present several numerical experiments to demonstrate the robustness of the proposed BBC algorithm. All experiments were conducted on a computer with an Intel Core i7-4790U

Algorithm 2 Root node

procedure RootNode
  CUTS ← $\emptyset$
  Continue ← TRUE
  while Continue **do**
    Continue ← FALSE
    $\bar{x}, \bar{z}$ ← Values of variables x and z when we solve the linear relaxation of (5), where the constraints (5d) are replaced with the constraints in set CUTS.
    for each O/D pair (i, j) **do**
      if the type of coverage is binary **then**
        for each $l \in L \setminus \{0\}$ with $\bar{z}^l_{ij} > 0$ **do**
          $S \leftarrow \text{MINCUT}(G^l_{ij})$
          if $\sum_{h \in H: k_h \notin S} \bar{x}_{ih} + \sum_{h \in H: m_h \in S} \bar{x}_{jh} < \bar{z}^l_{ij}$ **then**
            Continue ← TRUE
            Add $z^l_{ij} \leq \sum_{h \in H: k_h \notin S} x_{ih} + \sum_{h \in H: m_h \in S} x_{jh}$ to CUTS.
          end if
        end for
      else
        Solve the DS Formulation (15)
        if DS is unbounded **then**
          Continue ← TRUE
          $(\pi^*_{ijl}, \theta^*_{ijk}, \rho^*_{ijm})$ ← extreme ray that violates (15a)
          Add $\sum\limits_{l \in L} z^l_{ij} \pi^*_{ijl} + \sum\limits_{k \in H} x_{ik} \theta^*_{ijk} + \sum\limits_{m \in H} x_{jm} \rho^*_{ijm} \geq 0$ to NMP.
        end if
      end if
    end for
    if Continue == FALSE **then**
      Delete from CUTS all constrains whose dual value is null.
    end if
  end while
  Return CUTS.
end procedure

Binary coverage criterion	Partial coverage criterion
$f(c_{ijkm}) = \begin{cases} 1 & \text{if } c_{ij}^{km} \leq \beta \\ 0 & \text{otherwise} \end{cases}$	$f(c_{ijkm}) = \begin{cases} 1 & \text{if } c_{ij}^{km} \leq 0.75\beta \\ 0.75 & \text{if } 0.75\beta < c_{ij}^{km} \leq (0.8)\beta \\ 0.5 & \text{if } (0.8)\beta < c_{ij}^{km} \leq (0.85)\beta \\ 0.25 & \text{if } (0.85)\beta < c_{ij}^{km} \leq (0.9)\beta \\ 0 & \text{otherwise} \end{cases}$

Table 2: Function f for binary and partial coverage criteria.

processor (3.60 GHz CPU) and 16 GB of RAM. The operating system used was Ubuntu v18.04.2. The algorithm was implemented in Julia language v1.5.4, using the JuMP package v0.18.6. The management of the branch-and-bound tree and the LP resolutions were handled by CPLEX v12.10. We set $\epsilon = 0.02$ for all experiments. Additionally, the resolution of the minimum (i, j)-cut problems (line 5 of Algorithm 1) was performed using a specialized algorithm available in the *SparseMaxFlowMinCut* package (`https://github.com/artalvpes/SparseMaxFlowMinCut/`).

The remainder of this section is organized as follows. Section 5.1 describes how the instances tested in this study were generated, while Section 5.2 presents a comparison between the proposed algorithm and the formulations proposed by Stančić et al. (2022), Janković et al. (2017), and Peker and Kara (2015), for the respective problems UrApHMCP, UMApHMCP, and USApHMCP.

5.1. Instances generation

The instances used in this paper were generated based on two data sets frequently used for hub location problems: CAB (O'kelly, 1986) and AP (Ernst and Krishnamoorthy, 1996).

The CAB data set is derived from the Civil Aeronautics Board Survey of 1970 air passenger travel data in the United States. It provides passenger flows and distances between 25 cities ($|N| = 25$). The AP data set originates from Australian post office facilities. The original set contains flows and distances between 200 vertices, but we use smaller sets with 25 and 50 vertices, referred to as AP25 and AP50, respectively.

For all sets, each vertex is also a potential hub ($H = N$), and the calculation of the costs c_{ij}^{km} (Equation (1)) is based on the distance matrix, which is symmetric. Thus, we aggregate the O/D pairs (i, j) and (j, i) into a single pair with flow $w_{ij} + w_{ji}$. Additionally, we divide each O/D pair flow by the total sum of flows.

Following Peker and Kara (2015), Janković et al. (2017), and Stančić et al. (2022), we set $L = \{0, 1\}$ and $L = \{0, 0.25, 0.5, 0.75, 1\}$ for the binary and partial coverage criteria, respectively. The functions f for the binary and partial coverage criteria are described in Table 2. These functions use the constant β. For the CAB instances, the values of β were taken from Peker and Kara (2015). For the AP instances, we set the value of β to be equal to the corresponding solution value for the uncapacitated multiple allocation p-hub center problem (Ghaffarinasab, 2020).

For all experiments, we set $\chi = \delta = 1$ and $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$. All data sets used in this paper are online available at https://github.com/mcroboredo/Hub-Location-Instances

5.2. Comparison with the literature

We now present a comparison between the proposed algorithm and state-of-the-art exact literature formulations (2), (3), and (4) for the respective problems UrApHMCP, UMApHMCP, and USApHMCP. For each of the three problems, in addition to the proposed implementation of Algorithm (1), we also execute a variation where the proposed formulation (5) is replaced with the corresponding state-of-the-art exact formulation from the literature. For this, we made several modifications to the literature formulations. First, we replaced the variables z_{ij} with z_{ij}^l and added the constraints in (5c). Furthermore, for the formulations (2) and (3), we replaced the objective function (2a) with (5a) and modified the constraints (2h) to $z_{ij}^l \leq \sum_{k \in H} w_{ijkl}, \forall i, j \in N, l \in L$. For formulation (4), we replaced the objective function (4a) with (5a) and modified the constraints (4d) to $z_{ij}^l \leq \sum_{k \in H : a_{ijkm}^l = 1} x_{ik} + \lambda_{ij} (1 - x_{jm}), \forall i, j \in N, m \in H, l \in L$.

For a fair comparison, all formulations and algorithms were executed on the same machine.

Tables 3, 4, and 5 summarize the comparison for the UrApHMCP, UMApHMCP, and USApHMCP instances, respectively. In these tables, we present average statistics where the instances are grouped according to the type of coverage (binary or partial) and the data set (CAB, AP25, AP50). An online supplementary material is provided with detailed statistics for each individual instance. For each group of instances, the column headings in Tables 3, 4, and 5 are defined as follows:

- $\#Inst$: the number of instances in the group.
- *#opt* : the number of optimally solved instances within 7200 seconds of execution.
- Gap_0 : the average relative percentage difference between the upper bound at the root node and the optimal solution cost. For the state-of-the-art literature formulations, the upper bound at the root node is obtained by solving their linear relaxation. In Algorithm 1, this value is obtained by solving the procedure `RootNode`.
- $T_0(s)$: the total execution time in seconds to obtain the upper bound at the root node.
- Gap_0^{cplex} : the average relative percentage difference between the upper bound at the root node, given by the CPLEX solver, and the optimal solution cost.
- $T(s)$: the total computational time of execution in seconds. This time includes $T_0(s)$. We stop any experiment whose $T(s)$ reaches 7200 seconds.

In Tables 3, 4, and 5, we present only a single column for Gap_0 and a single column for $T_0(s)$ when referring to Algorithm 1. The reason for this is that we execute the `RootNode` procedure only with the proposed Formulation 5, as this procedure takes a considerable amount of time when considering the literature formulations.

Regarding the `RootNode` procedure, two implementation details are important to highlight. First, we add 50 cuts with the largest violation values per loop iteration to avoid the linear relaxation becoming difficult to solve, without this necessarily resulting in a significant improvement in the upper bound. Second, we stop the procedure after 10 consecutive iterations with an upper bound reduction smaller than 10^{-6}.

From the results in Tables 3, 4 and 5, one can see that the the proposed algorithm with the proposed formulation outperforms the other approaches for almost all instances. The only exceptions occur for small UMApHMCP instances with 25 nodes and partial coverage and for small USApHMCP instances with binary coverage from CAB25 data set. Moreover, the proposed algorithm with the proposed formulation optimally solved all 336 instances.

6. Conclusions

In this paper, we propose an exact algorithm for the UrApHMCP, UMApHMCP, and USApHMCP with binary and partial coverage criteria. The proposed algorithm is a branch-and-bound-and-cut (BBC) algorithm, implemented in a modern way where only a single tree search is considered, and the Benders cuts are generated on demand during the branch-and-cut process. Furthermore, we show that the dual subproblem is equivalent to a min-cut problem in certain special cases.

We present several computational experiments to compare the performance of three algorithms for each problem addressed in this paper: the state-of-the-art exact literature formulation, the proposed algorithm with the literature formulation, and the proposed BBC algorithm. The results demonstrate that the proposed algorithm with the proposed formulation outperforms the other approaches, especially for large instances.

For future research, a natural direction is to develop similar algorithms for other hub location problems (HLPs).

References

Button, K. (2002). Debunking some common myths about airport hubs. *Journal of Air Transport Management*, 8(3):177 – 188.

Campbell, J. F. (1994). Integer programming formulations of discrete hub location problems. *European Journal of Operational Research*, 72(2):387–405.

Contreras, I. and OKelly, M. (2019). Hub location problems. In *Location science*, pages 327–363. Springer.

								Algorithm (1)							
			State-of-art Lit. Formulation							Adapted Lit. Formulation			Proposed Formulation (5)		
Type of	Data			Average				Average			Average			Average	
Coverage	Set	$\#Inst.$	$\#opt$	Gap_0	$T_0(s)$	Gap_0^{cplex}	$T(s)$	Gap_0	$T_0(s)$	$\#opt$	Gap_0^{cplex}	$T(s)$	$\#opt$	Gap_0^{cplex}	$T(s)$
Binary	CAB25	24	24	4.1	1.5	1.6	11.3	1.5	3.0	24	1.1	23.1	24	1.1	7.0
	AP25	24	24	3.6	1.8	1.9	14.2	1.7	2.8	24	1.2	26.9	24	1.3	7.4
	AP50	24	2	1.7	482.1	1.5	6980.5	1.0	89.6	0	0.9	7200.0	24	0.9	265.8
Partial	CAB25	24	24	3.0	8.1	1.1	37.2	0.9	9.8	24	0.7	56.2	24	0.7	22.9
	AP25	24	24	2.1	11.1	1.1	58.3	0.8	9.9	24	0.6	71.6	24	0.6	29.4
	AP50	24	0	-	-	1.0	7200.0	0.5	251.3	0	0.5	7200.0	24	0.5	581.3

Table 3: Comparison with the literature for UrApHMCP instances.

Type of Coverage	Data Set	$\#Inst.$	State-of-art Lit. Formulation					Algorithm (1)							
				Average				Average		Adapted Lit. Formulation			Proposed Formulation (5)		
											Average			Average	
			$\#opt$	Gap_0	$T_0(s)$	Gap_0^{cplex}	$T(s)$	Gap_0	$T_0(s)$	$\#opt$	Gap_0^{cplex}	$T(s)$	$\#opt$	Gap_0^{cplex}	$T(s)$
	CAB25	16	16	3.9	1.6	1.3	6.5	1.4	2.6	16	1.2	11.5	16	1.2	4.9
Binary	AP25	16	16	3.6	1.6	1.6	7.5	1.6	2.6	16	0.7	32.0	16	1.1	5.3
	AP50	16	16	1.7	127.1	1.4	1093.3	0.9	28.0	16	0.8	1353.2	16	0.8	66.0
	CAB25	16	16	2.9	4.6	0.7	13.9	0.7	8.3	16	0.7	29.1	16	0.5	16.4
Partial	AP25	16	16	2.1	5.3	0.9	15.6	0.6	9.3	16	0.7	32.0	16	0.4	15.9
	AP50	16	5	1.0	652.5	0.8	5996.7	0.4	213.6	14	0.3	4733.5	16	0.3	395.6

Table 4: Comparison with the literature for UMApHMCP instances.

Type of Coverage	Data Set	$\#Inst.$	State-of-art Lit. Formulation					Algorithm (1)							
										Adapted Lit. Formulation			Proposed Formulation (5)		
				Average				Average			Average			Average	
			$\#opt$	Gap_0	$T_0(s)$	Gap_0^{cplex}	$T(s)$	Gap_0	$T_0(s)$	$\#opt$	Gap_0^{cplex}	$T(s)$	$\#opt$	Gap_0^{cplex}	$T(s)$
Binary	CAB25	16	16	7.0	1.0	3.6	7.4	1.8	3.4	16	1.1	11.1	16	1.4	7.5
	AP25	16	16	6.6	0.9	4.8	18.5	2.7	3.7	16	1.8	13.5	16	2.3	11.5
	AP50	16	8	3.0	61.0	2.6	4755.2	1.4	111.8	16	1.3	643.1	16	1.4	400.5
Partial	CAB25	16	16	5.2	1.0	5.0	109.3	1.1	12.4	16	0.8	39.8	16	0.9	31.7
	AP25	16	16	4.0	0.9	3.8	288.9	1.4	14.1	16	1.1	53.4	16	1.2	42.0
	AP50	16	1	2.1	197.7	2.1	7066.3	0.6	473.5	16	0.6	1863.7	16	0.6	1015.1

Table 5: Comparison with the literature for USApHMCP instances.

de Araújo, A. C. A., Roboredo, M. C., Pessoa, A. A., and Pereira, V. (2020). Exact methods for the discrete multiple allocation $(r|p)$ hub-centroid problem. *Computers & Operations Research*, 116:104870.

Ernst, A. T. and Krishnamoorthy, M. (1996). Efficient algorithms for the uncapacitated single allocation p-hub median problem. *Location science*, 4(3):139–154.

Errico, F., Crainic, T. G., Malucelli, F., and Nonato, M. (2017). A benders decomposition approach for the symmetric tsp with generalized latency arising in the design of semiflexible transit systems. *Transportation Science*, 51(2):706–722.

Farahani, R. Z., Hekmatfar, M., Arabani, A. B., and Nikbakhsh, E. (2013). Hub location problems: A review of models, classification, solution techniques, and applications. *Computers & Industrial Engineering*, 64(4):1096–1109.

Gendron, B., Scutellà, M. G., Garroppo, R. G., Nencioni, G., and Tavanti, L. (2016). A branch-and-benders-cut method for nonlinear power design in green wireless local area networks. *European Journal of Operational Research*, 255(1):151–162.

Ghaffarinasab, N. (2020). A highly efficient exact algorithm for the uncapacitated multiple allocation p-hub center problem. *Decision Science Letters*, 9(2):181–192.

Ghaffarinasab, N. and Kara, B. Y. (2019). Benders decomposition algorithms for two variants of the single allocation hub location problem. *Networks and Spatial Economics*, 19(1):83–108.

Ghaffarinasab, N., Motallebzadeh, A., Jabarzadeh, Y., and Kara, B. Y. (2018). Efficient simulated annealing based solution approaches to the competitive single and multiple allocation hub location problems. *Computers & Operations Research*, 90:173–192.

Hwang, Y. H. and Lee, Y. H. (2012). Uncapacitated single allocation p-hub maximal covering problem. *Computers & Industrial Engineering*, 63(2):382–389.

Ishfaq, R. and Sox, C. R. (2011). Hub location allocation in intermodal logistic networks. *European Journal of Operational Research*, 210(2):213 – 230.

Janković, O., Mišković, S., Stanimirović, Z., and Todosijević, R. (2017). Novel formulations and VNS-based heuristics for single and multiple allocation p-hub maximal covering problems. *Annals of Operations Research*, 259(1-2):191–216.

Janković, O. and Stanimirović, Z. (2017). A general variable neighborhood search for solving the uncapacitated r -allocation p -hub maximal covering problem. *Electronic Notes in Discrete Mathematics*, 58:23–30.

Kara, B. Y. and Tansel, B. C. (2003). The single-assignment hub covering problem: Models and linearizations. *Journal of the Operational Research Society*, 54(1):59–64.

Kim, H. and O'Kelly, M. E. (2009). Reliable p-hub location problems in telecommunication networks. *Geographical Analysis*, 41(3):283–306.

Li, X. and Aneja, Y. P. (2020). A branch-and-benders-cut approach for the fault tolerant regenerator location problem. *Computers & Operations Research*, 115:104847.

Moreno, A., Munari, P., and Alem, D. (2019). A branch-and-benders-cut algorithm for the crew scheduling and routing problem in road restoration. *European Journal of Operational Research*, 275(1):16–34.

O'kelly, M. E. (1986). A quadratic integer program for the location of interacting hub facilities. *European journal of operational research*, 32(3):393–404.

Peker, M. and Kara, B. Y. (2015). The p-hub maximal covering problem and extensions for gradual decay functions. *Omega*, 54:158–172.

Qu, B. and Weng, K. (2009). Path relinking approach for multiple allocation hub maximal covering problem. *Computers & Mathematics with Applications*, 57(11):1890–1894.

Silva, M. R. and Cunha, C. B. (2017). A tabu search heuristic for the uncapacitated single allocation p -hub maximal covering problem. *European Journal of Operational Research*, 262(3):954–965.

Stančić, O., Stanimirović, Z., Todosijević, R., and Mišković, S. (2022). Mathematical formulations and solution methods for the uncapacitated r-allocation p-hub maximal covering problem. *Discrete Optimization*, 43:100672.

Weng, K., Yang, C., and Ma, Y. (2006). Two artificial intelligence heuristics in solving multiple allocation hub maximal covering problem. In *International Conference on Intelligent Computing*, pages 737–744. Springer.